Flexural Vibration of Gravity-Stabilized, Structurally Damped, Large Flexible Satellites

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The stability and response of the in-plane flexural motion of gravity-stabilized, structurally damped, large flexible satellites in circular and eccentric orbits are analyzed. Employing the method of strained parameters, the stability analysis shows that the critical damping is sensitive to the satellite inertia ratio. It is noted that the orbital eccentricity excites motion in several high and low frequencies and generally decreases the stability of the system. An analytical expression for flexural response is obtained using the multiple-scales technique. Finally, stability charts and response are presented for a range of the system parameters.

Nomenclature

$A_I(\cdot)$	= modal amplitude of each mode
$A_{IN}(\cdot)$	= Nth perturbation in A_I
a_N, b_N	= constants of motion; bar indicates complex conjugate
c	= nondimensional disturbance parameter in pitch
e	= orbital eccentricity
I_x, I_y, I_z	= moments of inertia about principal axes x, y, z
i	= $\sqrt{-1}$
K, K_I	= $(I_y - I_x)/I_z$ and $\sqrt{3K}$, respectively
M	= modal mass
p	= modal frequency, nondimensionalized with respect to mean orbital rate
p_N, p_{Ne}	= Nth perturbation in p for circular and elliptic orbit, respectively
$R; R_x, R_y, R_z$	= distance of the Earth's center to the satellite mass center; its components
t	= time
ζ, ζ_c	= modal damping factor, critical damping
θ	= true anamoly
θ_I, θ_{I0}	= pitch angle, initial pitch
$\omega_x, \omega_y, \omega_z$	= components of body angular rate
τ	= $K_1\theta + d_2$

Dots and primes indicate differentiation with respect to t and θ (or τ), respectively.

Introduction

DURING the past 15 years there has been an increasing trend toward building very large flexible spacecraft. Associated with this trend has been the generation of a massive literature on the dynamics and control of flexible systems. The early work on the subject is well surveyed by Modi.[1] Most of the studies idealize the system as an assemblage of rigid bodies or a rigid body with flexible appendages. For many systems of the future, these idealizations may not be adequate. One would have to treat the entire system as elastic. Several approaches are now available to derive continuum models of large discrete structures.[2,3] Treating the system as an elastic continuum, this paper deals with in-plane flexural vibation of a large gravity-stabilized, structurally damped, flexible spacecraft undergoing pitch motion.

In an early study relevant to the present work, Chobotov[4] shows that the transverse and axial oscillations of a cable-connected system are represented by the Mathieu equation and that the stability criterion is a function of the cable frequency, orbital rate, and structural damping. Numerical solution[5] of the system indicates that structural damping is responsible for gradually, but very slowly, reducing the spin rate. Investigating the longitudinal and in-plane flexural vibrations of a beam-like satellite undergoing pitching oscillations, Ashley[6] concludes that the influence of infinitesimally small deformations on the moments of inertia is negligible for flexural vibration, but not for longitudinal vibrations. Modi and Brereton[7] derive modal equation for in-plane flexural vibrations of a flexible boom under solar heating and study the librational stability through numerical simulation. In a recent investigation, Kumar and Bainum[8] use the Mathieu stability charts and phase plane methods to assess stability of undamped in-plane flexural motion in circular orbit. Although their study succeeds in describing the gross behavior, it does not fully reveal the important effects of system parameters. In the present paper, an attempt is made to find analytical expressions leading to the stability conditions and response of the in-plane flexural motion of damped gravity-stabilized systems in circular as well as elliptic orbits. Using analytical solution for rigid-body modes and modal analysis, the equation of elastic motion are reduced to a set of linear equations with time-varying coefficients. They are analyzed for stability by application of the method of strained parameters. The resulting algebraic expressions show that the critical damping is sensitive to the vehicle inertia ratio. It is also noted that the oribtal eccentricity gives rise to both higher and lower frequency resonances. The stable response obtained using the multiple-scales technique is verified with the numerical solution of the equations of motion. The presentation includes stability charts in parametric space that should be useful to designers.

The present study complements work by the same authors on the stability of flexural and longitudinal vibrations of beam- and plate-shaped satellites in a torque-free environment.[9] This analysis is applicable to a number of satellite configurations, such as the gravity-stabilized solar power satellite,[10] geosynchronous passive communicator[11] etc.

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Analysis

The equations of motion of a flexible spacecraft modeled as an elastic continuum describing the orbital, attitude, and structural motions have been derived following the Newton-Euler approach by several authors.[6,8,12] For an elastic spacecraft moving around the spherical Earth under the influence of gravitational forces only, these equations of attitude and structural motion can be written as

$$(I_x + A_1^2 M)\dot{\omega}_x + (I_z - I_y + A_1^2 M)\omega_y\omega_z$$
$$= 3\Omega^2 [R_y R_z (I_z - I_y + A_1^2 M)/R^2] - 2\omega_x \dot{A}_1 A_1 M$$
$$I_y \dot{\omega}_y + (I_x - I_z)\omega_z\omega_x = 3\Omega^2 [R_z R_x (I_x - I_z)/R^2]$$
$$(I_z + A_1^2 M)\dot{\omega}_z + (I_y - I_x + A_1^2 M)\omega_x\omega_y$$
$$= 3\Omega^2 [R_x R_y (I_y - I_x - A_1^2 M)/R^2] - 2\omega_z \dot{A}_1 A_1 M$$
$$\ddot{A}_1 + 2\zeta\omega\dot{A}_1 + [\omega^2 - \omega_x^2 - \omega_z^2 + \Omega^2 - 3\Omega^2 R_y^2/R^2] A_1 = 0 \quad (1)$$

where $\Omega^2 = GM_E/R^3$ in which G is the gravitational constant and M_E the mass of the Earth.

The last equation describes the flexural vibration in the orbital plane for any one mode, as there exists little coupling among the modes. Since both structural and attitude motion have negligible effects on the orbital motion, it is decoupled and described by the Keplarian relation. Equation (1) clearly indicates that the effect of vibration on the vehicle moment of inertia is of second order and the equations are decoupled when this effect is neglected. Both Refs. 6 and 8 conclude that this is indeed the case when the entire system undergoes flexural vibrations. A number of studies on rigid-body dynamics of gravity-stabilized systems also indicate that the pitch motion is the most important one and that it can occur decoupled from roll and yaw motion. Therefore, considering planar motion only, the equations of pitch librations and flexural motion reduce to

$$\dot{\omega}_z = 3K\Omega^2 R_x R_y / R^2 \quad (2a)$$

$$\ddot{A}_1 + 2\zeta\omega\dot{A}_1 + (\omega^2 - \omega_z^2 + \Omega^2 - 3\Omega^2 R_y^2/R^2) A_1 = 0 \quad (2b)$$

Figure 1 schematically shows these motions for an arbitrarily shaped, large flexible satellite. It may be noted that Eqs. (2) are also applicable to multimode vibration because modal coupling is negligible. Equation (2a) can be solved for the given orbit and vehicle inertia ratio. In the sequel, stability and response of the flexural vibration described by Eq. (2b) is studied. It is observed that it is a linear damped time-varying system whose closed-form solution is not easily obtainable.

Flexural Stability in Circular Orbit

For small pitching oscillations of the gravity-stabilized space systems in circular orbit, Eq. (2a) reduces to a simple harmonic equation that can be solved to give

$$\theta_1 = d_1 \sin\tau \quad (3)$$

where

$$d_1 = (\theta_{10}^2 + \theta_{10}'^2/3K)^{1/2} \text{ and } d_2 = \tan^{-1}(\theta_{10} K_1/\theta_{10}')$$

Substitution of Eq. (3) into Eq. (2b) yields, after a change of variable from θ to τ

$$\mathrm{d}^2 A_1/d\tau^2 + (2\zeta p/K_1)\mathrm{d}A_1/\mathrm{d}\tau + [2Kp^2 - (1+K)c^2$$
$$-4Kc\cos\tau + (1-K)c^2\cos(2\tau)]A_1/6K^2 = 0 \quad (4)$$

It may be noted that Eq. (4) is a little different from that derived in Ref. 8. In Ref. 8, the contribution from $\theta_1'^2$ is retained but that from $\sin^2\theta_1$ is neglected, although both are of the same order. The results obtained, therfore, may not be quite correct. Equation (4), which represents the in-plane flexural vibration of a gravity-stabilized spacecraft with an initial pitch disturbance c, is a linear second-order differential equation with periodic coefficients. Stability of systems governed by this Whittakar equation can be established by direct application of the Floquet theory. However, this calls for extensive computation.

A common practice is to study stability under a small disturbance. The method of strained parameters, one of several perturbation techniques,[13] is an efficient approach to establish stability of the present system. The method requires expansion of the solution and the system frequency in a power series of the perturbative parameter. The coefficients are determined by eliminating secular terms. In practice, it is sufficient to truncate the series to the second-order term. Accordingly, the following expansions are assumed for the subsequent analysis:

$$A_1 = A_{10} + cA_{11} + c^2 A_{12}$$

$$p^2/3K - (1+K)c^2/6K^2 = p_0^2 + cp_1 + c^2 p_2$$

or

$$p/K_1 = p_0 + p_1 c/2p_0 + [p_2 + (1+K)/6K^2]c^2/2p_0 \text{ for } p_0 \neq 0 \quad (5)$$

Granted that these expansions are valid, one is interested in the effects of a small amount of structural damping i.e., the first-order damping represented by $\zeta = \zeta_1 c$.

Substituting Eq. (5) into Eq. (4) and equating the coefficients of equal powers of c to zero, one obtains a set of linear constant-coefficient, second-order differential equations. The equation corresponding to c^0 coefficient is solved to yield

$$A_{10} = a_0 \cos(p_0\tau) + b_0 \sin(p_0\tau) \quad (6)$$

where a_0 and b_0 are constants depending upon the initial conditions of flexture. This is the zeroth-order solution. Using this in the equation corresponding to c^1, one can determine A_{11}. For a uniformly valid expansion to exist at the primary zone of resonance, $p_0 = 1/2$, the terms giving rise to secular response in A_{11} must vanish. Omitting the algebraic details, this leads to the following expression, which is valid on the transition curves:

$$p_1 = \pm(1/9K^2 - 4\zeta_1^2 p_0^4)^{1/2} \text{ and } a_0 = (1/3K + p_1)b_0/(2\zeta_1 p_0^2) \quad (7a)$$

With these, the equation for A_{11} is solved to give

$$A_{11} = -\{a_0\cos[(p_0+1)\tau] + b_0\sin[(p_0+1)\tau]\}$$
$$\div [3K(1+2p_0)] - a_1\cos(p_0\tau) + b_1\sin(p_0\tau) \quad (7b)$$

Following the same procedure, one determines A_{12} from the equation obtained by comparing coefficients of c^2 and using A_{10} and A_{11}. Invoking the condition of periodicity of the solution A_{12} at $p_0 = 1/2$, yields

$$p_2 = p_1^2 - 1/6K^2 - \zeta_1^2/4 \quad (8)$$

Combining Eqs. (5), (7), and (8), we obtain the relation for transition curves near the primary zone of resonance ($p_0 = 1/2$) to the second order, as

$$p = K_1[1/2 \pm (c^2/9K^2 - \zeta^2/4)^{1/2} + (3K+2)c^2/18K^2 - \zeta^2/2] \quad (9)$$

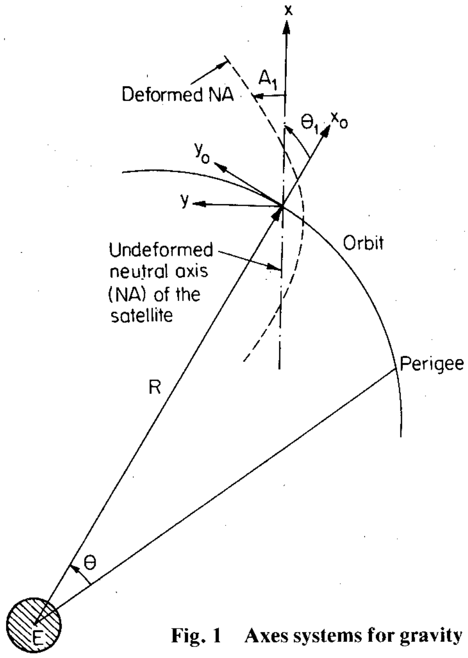

Fig. 1 Axes systems for gravity gradient satellite.

The right-hand side of Eq. (9) will have real values only if $c \geq 1.5K\zeta$. This implies that the maximum level of disturbance c_{max} that can be tolerated by the structure with stable flexural motion near the primary resonance is

$$c_{max} = 1.5K\zeta \tag{10}$$

A more important quantity is the critical damping necessary to avoid vibrational instability for a given level of disturbance. This can be derived as

$$\zeta_c = c/(1.5K) \tag{11}$$

Note that it is directly proportional to the disturbance level and inversely proportional to the inertia ratio.

For $p_0 \neq \frac{1}{2}$, it can be shown that the determinant of the system formed for a_0 and b_0 in order to eliminate the secular terms can never be equal to zero for any real values of ζ or p_0. This implies that there does not exist a stability boundary. Thus, damping of the first order removes instability regions near all other resonance conditions that may be present otherwise. Incorporation of damping higher than first order will at best show up transition curves defined by Eq. (9). Therefore, it remains to investigate the effects of the second-order damping given by $\zeta = c^2\zeta_1$. Following the same steps as before, one obtains, for $p_0 = 1$,

$$p_1 = 0 \text{ and } p_2 = 2/27K^2 \pm [(3K+1)^2/(1296K^4) - 4\zeta_1^2]^{1/2} \tag{12}$$

Hence the stability boundaries near $p_0 = 1$ are given by

$$p = K_1[1 + (13+9K)c^2/(108K^2) \pm [3K+1)^2c^2/(1296K^4) - 4\zeta^2]^{1/2} c/2] \tag{13}$$

and the maximum tolerable disturbance level and the critical damping at this resonance are given by

$$c_{max} = 72K^2\zeta/(3K+1) \tag{14}$$

$$\zeta_c = (3K+1)c/(72K^2) \tag{15}$$

As expected, the critical damping requirement for the second resonance is less than that for the primary resonance. Proceeding further it is easy to show that the second-order damping removes all resonances for $p_0 \geq 2$ in the case of small disturbances.

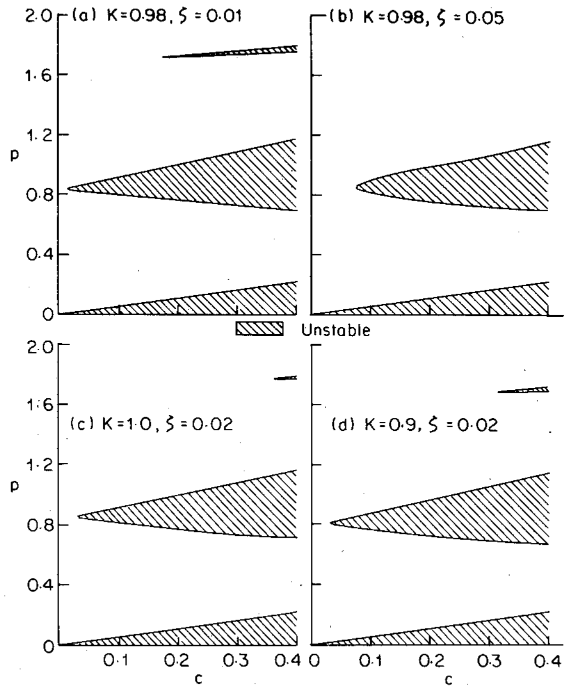

Fig. 2 Stability of flexural vibration in circular orbits.

Attention is now drawn to find instability near the zone of $p_0 = 0$. To this end, let $p/K_1 = p_2 c$. With the first-order damping, the stability boundary is now found to be

$$p = (\tfrac{1}{2} - 1/6K)^{1/2} c \tag{16}$$

Equation (16) implies that stability near $p_0 = 0$ is not affected by the presence of structural damping. The same conclusion is reached even when the analysis includes full damping. Discussion of the above transition curves and instability zones is presented in a later section.

Flexural Response in Circular Orbit

Study of dynamic response of a flexible spacecraft is important. The technique used to find the stability domain gives solutions only on the transition curves. To find the response of the damped system within the stable region away from the boundary, the multiple-scales technique[13] is chosen. For this purpose, let

$$\delta^2 = p^2/3K - (1+K)c^2/6K^2, \qquad \zeta = \zeta_1 c$$

and let the expansion representing the response be the following function of different time scales T_N:

$$A_1(\tau, c) = A_{10}(T_0, T_1, T_2) + cA_{11}(T_0, T_1, T_2) + c^2 A_{12}(T_0, T_1, T_2) \tag{17}$$

where $T_N = c^N \tau$, in which $N = 0,1,2,...$ are the new independent variables. Since the expansion up to second order in c is considered to give a sufficiently accurate response, the derivatives w.r.t. τ can be written as

$$\frac{d}{d\tau} = D_0 + cD_1 + c^2 D_2, \qquad D_N = \frac{\partial}{\partial T_N}$$

$$\frac{d^2}{d\tau^2} = D_0^2 + 2cD_0D_1 + c^2(D_1^2 + 2D_0D_2) \tag{18}$$

Substituting Eqs. (17) and (18) into Eq. (4) and equating the coefficients of equal powers of c to zero, a set of linear constant-coefficient, second-order differential equations is obtained. For δ away from zero, the solution of the equation corresponding to the coefficient of c^0 can be conveniently written as

$$A_{10} = a_0(T_1,T_2)e^{i\delta T_0} + \bar{a}_0(T_1,T_2)e^{-i\delta T_0} \qquad (19)$$

where a_0 and $\bar{a}_0$, complex conjugates to each other, are to be determined as functions of T_1 and T_2 constraining A_{11} and A_{12} to be periodic in T_0. The right-hand side of the equation for A_{11}, obtained by comparing the coefficients of c, contains secular terms. Using Eq. (19), it is found that, for δ away from ½, the secular terms can be eliminated if,

$$a_0 = a_{01}(T_2)\exp(-\zeta_1 pT_1/K_1) \qquad (20)$$

With this, the particular solution for A_{11} can be written as

$$A_{11} = \exp(-\zeta_1 pT_1/K_1)\{-a_{01}\exp[i(\delta+1)T_0]/(2\delta+1) + a_{01}\exp[i(\delta-1)T_0]/(2\delta-1) + \bar{a}_{01}\exp[-i(\delta-1)T_0] \div (2\delta-1) - \bar{a}_{01}\exp[-i(\delta+1)T_0]/(2\delta+1)\}/3K \qquad (21)$$

Substituting Eq. (21) into the equation corresponding to the coefficients of c^2 and eliminating secular terms for the solution of A_{12} for δ away from unity, one gets

$$a_{01} = a\exp\{-i[\{2/[3K(4\delta^2-1)] + \zeta_1^2p^2\}T_2/(6K\delta) + b]\} \qquad (22)$$

where a and b are now real constants to be determined from the initial conditions. The particular solution of A_{12} is then found as

$$A_{12} = \exp(-\zeta_1 pT_1/K_1)\{E_1 a_{01}\exp[i(\delta+2)T_0] + E_2 a_{01}\exp[i(\delta-2)T_0] + E_2\bar{a}_{01}\exp[-i(\delta-2)T_0] + E_1\bar{a}_{01}\exp[-i(\delta+2)T_0]\} \qquad (23)$$

where

$$E_1 = \{1/[9(2\delta+1)] + (1-K)/12\}/[K^2(4\delta+4)]$$

$$E_2 = \{1/[9(2\delta-1)] - (1-K)/12\}/[K^2(4\delta-4)]$$

Combining Eqs. (17), (19), (21), and (23) and expressing the complex exponentials in the form of trigonometric functions, the stable solution of Eq. (4) for δ away from 0, ½, and 1 is given by

$$A_1 = a\exp(-\zeta p\tau/K_1)\{\cos(\omega_m\tau - b) - c\cos[(\omega_m+1)\tau - b] \div [3K(2\delta+1)] + c\cos[(\omega_m-1)\tau - b] \div (3K(2\delta-1)] + \{1/[3(2\delta+1)] + (1-K)/4\} \times c^2\cos[\omega_m+2)\tau - b]/[12K^2(\delta+1)] + \{1/[3(2\delta-1)] - (1-K)/4\}c^2\cos[(\omega_m-2)\tau - b]/\{12K^2(\delta-1)]\} \qquad (24)$$

where $\omega_m = \delta - \{2c^2/[3K(4\delta^2-1)] + \zeta^2p^2\}/6K\delta$, which can be termed as nondimensional mean frequency to the order of c^2. The response contains sub- and superharmonics having the dominant component of the mean frequency. It is also observed that the solution exhibits an exponentially decaying behavior. This is well known from the theory of linear damped systems. Before discussing stability and response of flexural motion in circular orbit, it is worthwhile completing the analysis for systems in elliptic orbits.

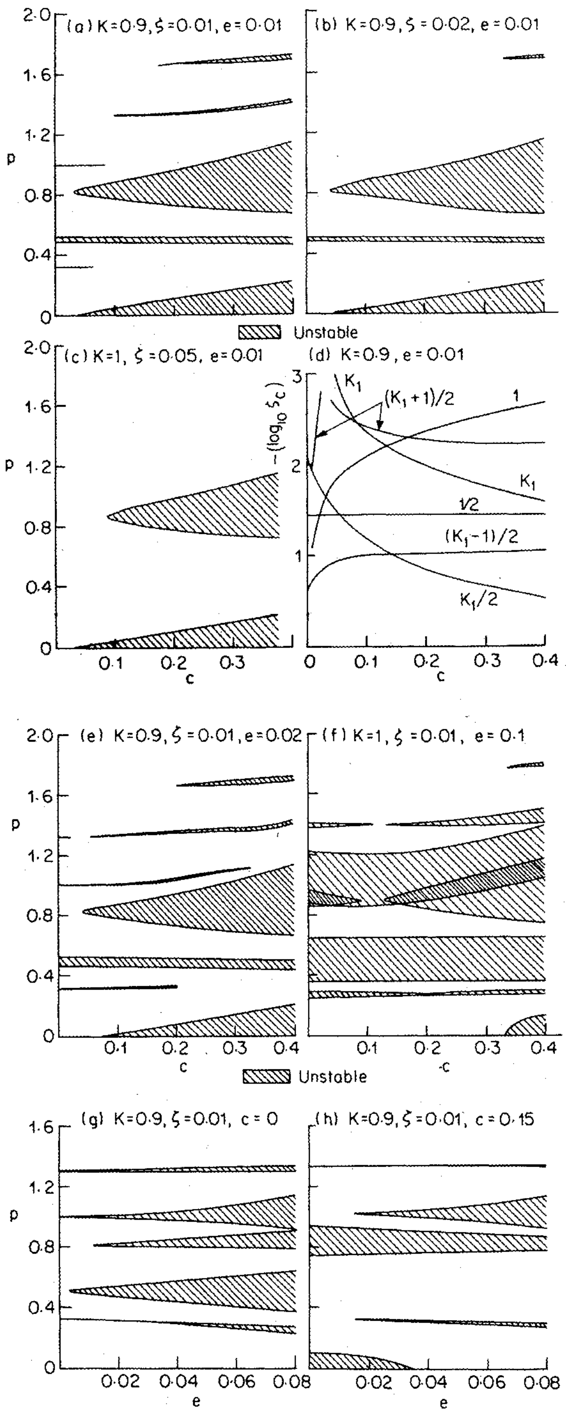

Fig. 3 Stability of flexural vibration in elliptic orbits.

Flexural Stability in Elliptic Orbits

It is well known that satellite librations in eccentric orbits are more complicated than those in circular orbits. Gyroscopic effects and eccentricity-induced forcing motion give rise to a highly nonlinear dynamical system. Having established the stability and response of in-plane flexural motion in circular orbits, a similar approach is adopted for elliptic orbits to investigate the effects of orbit eccentricity. For small-amplitude libration, Eq. (2a) can be reduced to the following well-known linearized equation of planar libration[14]:

$$(1+e\cos\theta)\theta_l'' - 2e\sin\theta\theta_l' + 3K\theta_l = 2e\sin\theta$$

Even for this simple equation there does not exist a closed-form solution. For small-eccentricity orbits, an approximate solution[14] obtained employing the Wentrel, Kramers, Boillouin and Jeffereys (WKBJ) method is given by

$$\theta_1 = F_1 \sin(K_1\theta + F_2) + eB_1F_1 \sin[(K_1+1)\theta + F_2] + eB_2F_1 \sin\theta - \sin[(K_1-1)\theta + F_2] + 2e[K_1 \sin\theta - \sin(K_1\theta)]/[K_1(3K-1)] \quad (25)$$

where

$$B_1 = -(1/6K + K_1 - 1/K_1 + 3/2)/4$$

$$B_2 = -(1/6K - K_1 + 1/K_1 + 3/2)/4$$

$$F_1^2 = \theta_{10}^2/f^2 + \theta_{10}'^2/3Kf^2$$

$$F_2 = \tan^{-1}\{[K_1 - e(1/K_1 + 5K_1/2)/2]\theta_{10}/f\theta_{10}'\}$$

$$f = 1 - e(5 - 1/3K)/4$$

Equation (25) shows that three additional frequency components are now introduced due to the orbital eccentricity. Neglecting terms of order e^2, Eq. (2b) can be written as

$$A_1'' + [2\zeta p - 2(2\zeta p\cos\theta + \sin\theta)e]A_1' + [(p^2 - 2\theta_1' - \theta_1'^2 - 3\theta_1^2) - (4p^2 + 1 - 3\theta_1^2)e\cos\theta]A_1 = 0 \quad (26)$$

Equation (26) represents the flexural vibration in the orbital plane of a flexible spacecraft moving in an orbit of small eccentricity and reduces to Eq. (4) for zero eccentricity. Considering only impulsive disturbances for pitch motion ($F_2 = 0$), the substitution of Eq. (25) into Eq. (26) and simplification lead to

$$A_1'' + 2[\zeta p - (2\zeta p\cos\theta + \sin\theta)e]A_1' + \{p^2 - c_1c^2 - c_2c\cos(K_1\theta) + c_3c^2\cos(2K_1\theta) + [c_4c - (c_5 + 4p^2)\cos\theta - c_6c\cos[(K_1-1)\theta] + c_7\cos(K_1\theta) - c_8c\cos[(K_1+1)\theta] - c_9c\cos(2K_1\theta)]e\}A_1 = 0 \quad (27)$$

where

$$c_1 = (1+K)/2Kf^2; \quad c_2 = 2/f; \quad c_3 = (1-K)/2Kf^2;$$

$$c_4 = 2(1+K)/Kf(3K-1); \quad c_5 = 3(1+K)/(3K-1);$$

$$c_6 (c_{61} + c_{62})/4K_1f(3K-1); \quad c_7 = 4/(3K-1);$$

$$c_8 = (-c_{61} + c_{62})/4K_1f(3K-1);$$

$$c_9 = 2(1-K)/Kf(3K-1);$$

$$c_{61} = (54K^3 - 9K^2 + 72K - 1)/3K;$$

$$c_{62} = (-45K^2 + 42K - 1)/K_1; \quad c = \theta_{10}'$$

Occurrence of frequencies other than K_1 due to the effect of eccentricity indicates that the flexural motion will now find many more resonance points than those in the case of circular orbits. It is also to be noted, although not pursued here, that the phase difference ($F_2 \neq 0$) due to the initial librations influences the stability and response of the flexural vibration.

Equation (27) represents an almost periodic time-varying system and Floquet theory cannot be applied directly. Reference 13 does not analyze such systems. For stability analysis, the method of strained parameters is again followed here. In addition to the expansions corresponding to Eq. (5), define $e = e_1c$ and substitute them in Eq. (27) to obtain a set of linear constant coefficient differential equations. p is defined as

$$p = p_{0e} + p_{1e}c/2p_0 + p_{2e}c^2/2p_0$$

As before, the stability boundaries are then obtained by establishing conditions to avoid secular terms in the solutions of these equations. Without going into the details of the algebra, the final results are presented below. Near the resonance $p_{0_e} = ½$, the transition curves are given by

$$p = (1-\zeta^2)/2 \pm \{9(1+K)^2e^2/[(3K-1)^2] - \zeta^2\}^{½}/2 + (3K+1)(1-K)c[c/2f - 2e \div (3K-1)]/[Kf(3K-1)] + (297K^3 - 81K^2 - 93K - 35)e^2/[8(3K-1)^3] \quad (28)$$

For small disturbances, this can be approximated to

$$p = ½ \pm \{9(1+K)^2e^2/[(3K-1)^2] - \zeta^2\}^{½}/2$$

Equation (28) yields the following formula for critical damping:

$$\zeta_c = 3(1+K)e/(3K-1) \quad (29)$$

The fact that ζ_c is independent of the disturbance level implies that this instability zone is primarily due to the forced libration in eccentric orbits.

Near the resonance $p_{0_e} = K_1/2$, the transition curves are given by

$$p = K_1(1-\zeta^2)/2 \pm \{4[c/f - 2e/(3K-1)]^2 - 9K^2\zeta^2\}^{½}/2K_1 + (3K+2)c[c/2f - 2e/(3K-1)]/(3KK_1f) + (243K^5 - 81K^4 + 162K^3 - 45K^2 + 33K + 4)e^2 \div [6KK_1(3K-1)^3] \quad (30)$$

Owing to its negligible contribution, the last term can be omitted. The critical damping is, therefore,

$$\zeta_c = \pm 2[c/f - 2e/(3K-1)]/3K \quad (31)$$

Hence, ζ_c depende upon K, c, and e; the eccentricity dependency arises mainly from the forced libration. It is noted that the damping requirement to stabilize the vibrations for a given level of disturbance is less than that required in the circular orbit. Such a conclusion is not possible if the phase difference of the initial libration is considered. For example, if $F_2 = \pi$, the correct expression for p is obtained by putting $c = -c$ and this gives a higher value of ζ_c. A unique feature of this instability zone is that for a spacecraft with a finite amount of damping, stability is insured only within the following range of disturbance level:

$$0 < [-3K\zeta/2 + 2e/(3K-1)] < c < [3K\zeta/2 + 2e/(3K-1)] \quad (32)$$

In the case of circular orbits, the lower limit reduces to zero. Consequently, the region of vibrational instability is expanded in the eccentric orbits.

Near the resonance $p_{0_e} = 1$, the stability boundary and critical damping are found to be

$$p = 1 \pm \{ (135K^2 + 6K - 1)^2 e^4 / [(3K-1)^2] - 64\zeta^2 c^2 \}^{1/2} / 4$$
$$+ (3K^2 - 5K - 4) [c/2f - 2e/(3K-1)] c / [2(3K-4)Kf]$$
$$+ (171K^3 - 246K^2 + 27K - 20) e^2 / [4(3K-1)^2 (3K-4)] \quad (33a)$$

$$\zeta_c = (135K^2 + 6K - 1) e^2 / [8(3K-1)c] \quad (33b)$$

For a given eccentricity ζ_c is inversely proportional to c. This peculiar property suggests a new type of instability domain not found in the case of the Hill/Mathieu equation. The instability is caused primarily by the forced libration in eccentric orbits and exists only for low levels of disturbances. At higher disturbance levels, the effect of damping dominates. Therefore, the lower the limit of disturbance beyond which stability is desired, the higher is the requirement for damping in a given orbit. Of course, as has been found earlier, first-order damping removes the instability in this region.

Near the resonance conditions $p_{0_e} = (K_I \pm 1)/2$, the transition curves are

$$p = (K_I \pm 1)/2 + [2K_I (1+K) \pm (5K+1)] [c/2f$$
$$- 2e/(3K-1)] c / [(1 + 6K \pm 3K_I) Kf]$$
$$\pm \{ [[162K^3 + 9K^2 + 1 \pm K_I (99K^2 - 6K - 1)] c / (12Kf)$$
$$- 4 [(9K^2 + 3) \pm 2K_I (3K-1)] e$$
$$\div (3K-1)]^2 e^2 / [3(3K-1)^2 K] - (K_I \pm 1)^4 \zeta^2 c^2 \}^{1/2}$$
$$\div [2(K_I \pm 1)] + [3(243K^4 + 9K^3 + 36K^2 + K + 3)$$
$$\pm 2K_I (81K^4 + 189K^3 - 45K^2 + 27K + 20)] e^2 / \{ 2(3K-1)^2$$
$$\times [K_I (21K + 2) \pm 3K(6K+7)] \} \quad (34)$$

In practice, the last term has negligible contribution and can be omitted. Critical damping required to avoid this instability zone can be obtained as

$$\zeta_c = \pm \{ [162K^3 + 9K^2 + 1 \pm K_I (99K^2 - 6K - 1)] e / (12Kf)$$
$$- 4 [9K^2 + 3 \pm 2K_I (3K-1)] e^2 / [(3K-1)c] \}$$
$$\div [(K_I \pm 1)^2 K_I (3K-1)] \quad (35)$$

Except for the first one, all $\pm$ signs follow the value of p_{0e} at resonance. ζ_c now has two components: one proportional to e and the other to e^2/c. If the phase difference of the libration is considered, ζ_c will increase. This instability zone does not occur in the case of circular orbit. Also, the first-order damping removes this instability zone. For a given set of e, ζ, and K, the range of c for which vibrational stability is available can be obtained from

$$c = 4 [(9K^2 + 3) \pm 2K_I (3K-1)] e^2 / [(3K-1) \{ [162K^3$$
$$+ 9K^2 + 1 \pm K_I (99K^2 - 6K - 1)] e / (12Kf)$$
$$\pm \zeta K_I (K_I \pm 1)^2 (3K-1)] \} \quad (36)$$

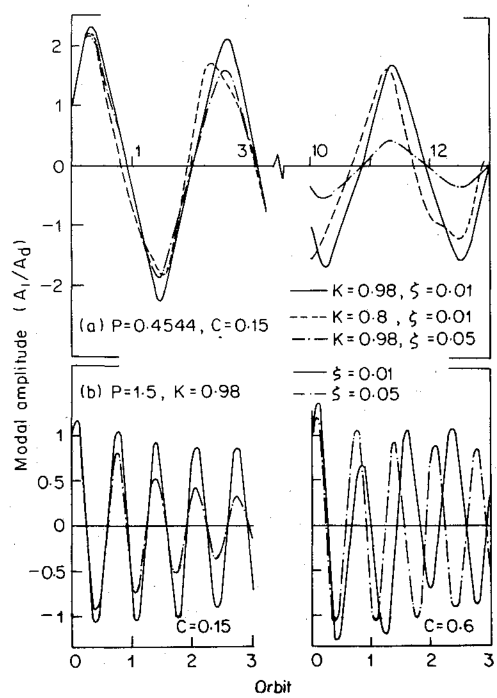

Fig. 4 Response of flexural vibration in circular orbits.

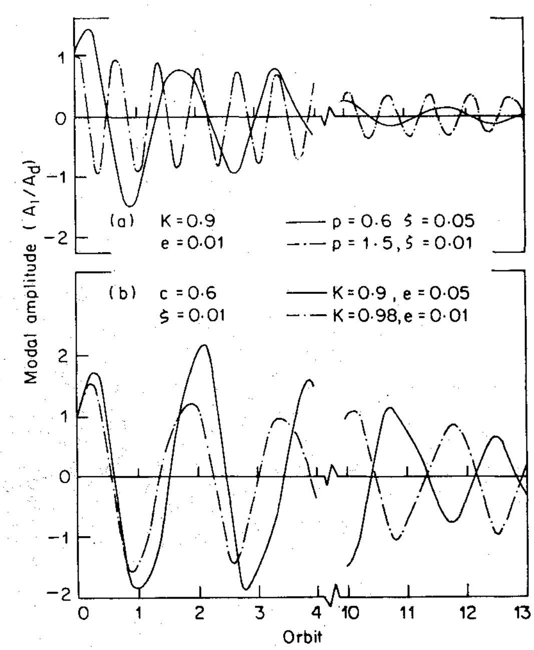

Fig. 5 Response of flexural vibration in elliptic orbits.

The plus sign in front of ζK_I gives the lower limit, while the minus sign gives the upper limit; other signs are chosen according to the resonance value of p_{0e}.

Near the resonance $p_{0e} = K_1$, the transition curves correspond to

$$p = K_I + (13 + 9K) [c/2f - 2e/(3K-1)] c / (18KK_I f) \pm \{ (3K+1)^2 c^4 / (4f^4) + 64e^4 / [(3K-1)^4] - 32(1+3K) e^3 c / [(3K-1)^3 f]$$
$$+ 12(3K+1)(K+1) e^2 c^2 / [(3K-1)^2 f^2] - 2(3K+1)^2 e c^3 / [(3K-1) f^3] - 1296K^4 \zeta^2 c^2 \}^{1/2} / (12KK_I)$$
$$+ [-1215K^3 + 378K^2 - 327K + 16 - 108KK_I (27K^2 - 6K - 1)] e^2 / [36KK_I (1 - 12K)(3K-1)^2] \quad (37)$$

The last expression and the second term under the square root are neglible. The critical damping for this zone is

$$\zeta_c = \{(3K+1)^2/(4f^4) + 64e^4/[(3K-1)^4c^4] - 32(1+3K)e^3/[(3K-1)^3fc^3]$$
$$+12(3K^2+6K-1)e^2/[(3K-1)^2f^2c^2] - 2(3K+1)^2e/[(3K-1)f^3c]\}^{1/2}c/(36K^2) \tag{38}$$

It is proportional to c and also a function of (e/c). The case of circular orbit instability can be seen to be a special case of that in the eccentric orbit. This instability zone too can be removed by incorporating damping of the first order.

Near $p_{0e}=0$ the stability boundary is found to be

$$p = \{(3K-1)c^2/(6Kf^2) - 2ec/(3Kf) + (27K^3-18K^2-45K-16)e^2/[6K(3K-1)^2]\}^{1/2} \tag{39}$$

The term containing e^2 can be neglected. As was the case for circular orbits, here too damping does not alter stability of vibration. However, one finds that if $c \leq 4fe/(3K-1)$ for a given e or for a given level of disturbance c if $e \geq (3K-1)c/4f$, there does not exist any instability. This, therefore, is the only case where orbital eccentricity is found to be aiding vibrational stability.

The instability zones discussed above are the only ones that occur in the case of elliptic orbits.

Flexural Response in Elliptic Orbits

Having established stability, analytical expressions for stable response are developed using the method of multiple scales. The multiple-time scales are defined as $\theta^N = c^N\theta$. Repeating the steps given in the case of circular orbit, the final result is found to be

$$A_l = a\exp(\zeta p\theta)\{\cos(\omega_m\theta - b) - e(c_5+4p^2+2p)\cos[(\omega_m+1)\theta-b]/(4p+2) + e(c_5+4p^2-2p)\cos[(\omega_m-1)\theta-b]/(4p-2)$$
$$-(c_2c-ec_7)\cos[(\omega_m+K_l)\theta-b]/(4pK_l+6K) + (c_2c-ec_7)\cos[(\omega_m-K_l)\theta-b]/(4pK_l-6K) + e\zeta p\sin[(\omega_m+1)\theta-b]$$
$$-e\zeta p\sin[(\omega_m-1)\theta-b] + (c_5+4p^2+2p)(c_5+4p^2+2p+2)e^2\cos[(\omega_m+2)\theta-b]/[16(2p+1)(p+1)] + e^2(c_5+4p^2-2p)$$
$$\times(c_5+4p^2-2p+2)e^2\cos[(\omega_m-2)\theta-b]/[16(2p-1)(p-1)] + X_1\cos[(\omega_m+K_l+1)\theta-b] - X_2\cos[(\omega_m+K_l-1)\theta-b]$$
$$+X_3\cos[(\omega_m-K_l+1)\theta-b] + X_4\cos[(\omega_m-K_l-1)\theta-b] + X_5\cos[(\omega_m+2K_l)\theta-b] + X_6\cos[(\omega_m-2K_l)\theta-b]\} \tag{40}$$

where the constants X are related to the coefficients of Eq. (40) as

$$X_1 = [(c_2c-ec_7)(c_5+4p^2+2p+2K_l)/(8pK_l+12K) + (c_2c-ec_7)(c_5+4p^2+2p)/(8p+4) - c_8c/2]e/(2pK_l+2p+3K+1+2K_l)$$

$$X_2 = [-(c_2c-ec_7)(c_5+4p^2-2p-2K_l)/(8pK_l+12K) + (c_2c-ec_7)(c_5+4p^2-2p)/(8p-4) + c_6c/2]e/(2pK_l-2p-2K_l+3K+1)$$

$$X_3 = [(c_2c-ec_7)(c_5+4p^2+2p-2K_l)/(8pK_l-12K) - (c_2c-ec_7)(c_5+4p^2+2p)/(8p+4) + c_6c/2]e/(2pK_l-2p-3K-1+2K_l)$$

$$X_4 = [(c_2c-ec_7)(c_5+4p^2-2p+2K_l)/(8pK_l-12k) + (c_2c-ec_7)(c_5+4p^2-2p)/(8p-4) + c_8c/2]e/(2pK_l+2p-3K-1-2K_l)$$

$$X_5 = [(c_2c-ec_7)^2/(4pK_l+6K) + (c_3c-ec_9)c]/(8pK_l+24K)$$

$$X_6 = [(c_2c-ec_7)^2/(4pK_l-6K) - (c_3c-ec_9)c]/(8pK_l-24K)$$

and the mean frequency of response ω_m is defined as

$$\omega_m = p - [\zeta^2p^2 + [(c_5+4p^2)(c_5+4p^2-2) + 4p^2]e^2/(8p^2-2) + (c_2c-ec_7)^2/(8p^2-6K) + (c_1c^2-ecc_4)]/(2p) \tag{41}$$

The response in the circular orbit is seen to be a special case of Eq. (40). Note that in this case δ is not defined. The present response contains components of 12 frequencies that are functions of librational frequencies. Presence of the sine function is indicative of the influence of gyroscopic motion due to eccentricity. As may be expected, structural damping is solely responsible for decaying vibration. Only amplitudes corresponding to the frequencies that are functions of K_l depend on the disturbance. The other frequencies, except the mean frequency, have amplitudes directly resulting from the eccentricity-induced forced libration. In spite of a complex combination of a set of cosine and sine functions in the response expression, one can, in principle, find the maxima and their time of occurrence by solving a transcendental equation. This is not pursued here.

Parametric Study

Stability of Flexural Vibration

The results of the foregoing stability analysis in circular orbit are presented in Fig. 2 on the p-c plane for certain ranges of parameters. Unlike the plots given in Ref. 8, the ones presented here lend themselves to a direct interpretation. The instability zones near frequencies $p=0$, $K_l/2$, and K_l are shown as three distinct regions separated by the stable regions. They are bounded by two curves, the upper one corresponding to the plus sign and the lower to the minus sign in the equations for the stability boundaries. Near the zero frequency, the lower boundary is given by the line $p=0$ because p cannot take negative values. The point of intersection of the lower and upper boundaries is strongly dependent on the inertia ratio and the damping factor. It establishes (except near the zero frequency instability zone) the stability of all structural modes for a level of disturbance. For example, a spacecraft with $K=0.98$ and $\zeta=0.01$ (Fig. 2a), all structural modes having frequencies $p \geq K_l/2$ are stabilized against disturbances of $c \leq 0.025$. If the structure's fundamental frequency is given by $p \geq K_l$, the maximum disturbance tolerable from Fig. 2a is $c=0.175$. High-frequency modes, therefore, enjoy high degrees of stability. One is led to the same conclusion if one examines the critical damping required to stabilize a particular mode. For $K=0.98$, $c_{max}=0.5$, the critical damping for modes having frequencies $p \simeq K_l$ is $\zeta_c=0.028$ and, this being less than $\zeta=0.05$, the instability region near $p=K_I$ is absent in Fig. 2b. As one approaches modes of lower frequency, the value of critical damping increases, ending with the absence of influence of damping on extremely low-frequency modes near $p \simeq 0$. A comparison of Figs. 2c and 2d indicates that decreasing K shifts the instability regions to lower frequencies. For a given space structure $p_\propto$ (orbital radius)[1,5], therefore, high-altitude orbits offer a greater chance of elastic stability for the satellite.

Stability charts (Fig. 3) of flexural vibration in elliptic orbit show up more instability zones in addition to those occurring near frequencies of 0, $K_I/2$, and K_I in circular orbit. These occur near the frequencies $(K_I-1)/2$, ½, $(K_I+1)/2$, and 1. Equations representing their boundaries are more complicated. Although their width is comparatively small, they make the structure unstable to a higher degree, narrowing the choice of frequencies of stable structural modes within the expected range of disturbance levels. For a satellite of $K=0.9$ and $e=0.01$, increasing ζ to 0.02 (Fig. 3b) removes the instability zone near the frequency $(K_I+1)/2$ and shortens the line near frequencies $(K_I-1)/2$ and 1. The line results from the very small width of the unstable region. The regions near $p=K_I$ and $p=K_I/2$ become more stable against disturbances. The unstable zone near $p=$ ½ remains practically unchanged. The region near $p=0$ does not depend on damping. Increasing ζ to 0.05 (Fig. 3c) removes all unstable regions except those near $p=K_I/2$ and 0.

It was shown earlier that the critical damping for all instability zones depends on the eccentricity. To see its dependence on the external disturbance, $-\log_{10}\zeta_c$ is plotted against c for $K=0.9$ and $e=0.01$ (Fig. 3d). In this case, for disturbance level $c>0.015$, the maximum value of ζ_c corresponds to that given for the instability zone near $p=K_I/2$. For $c<0.015$, $\zeta_{c\,\max}$ is given by the instability zone near $p=(K_I-1)/2$. Incorporation of structural damping more than the maximum critical damping stabilizes all structural modes. In general, as pointed out earlier, ζ_c for frequencies $K_I/2$, $(K_I-1)/2$, $(K_I+1)/2$, and K_I depend on the phase of the initial libration and the worst case would correspond to either 0 or π. This is not pursued further here.

The effect of eccentricity on stability can be observed by comparing Figs. 3a, 3e, and 3f. For higher values of eccentricity, stable gaps in the unstable regions near $p=(K_I+1)/2$, and $K_I/2$ are noticed. The instability zones near $p=0$ and K_I are also reduced. Since the instability near frequencies 1 and ½ is mainly caused by the eccentricity-induced librations, the corresponding unstable zones are widened for large eccentricity. The instability zones can overlap, modifying the stability boundaries (Fig. 3f).

Figures 3g and 3h present stability plots on the *p-e* plane. Generally, as eccentricity increases, the stable regions diminish rapidly. In the absence of any external disturbance, the modes near $p=0$ are never destabilized. Modes of other frequencies exhibit a trend of instability similar to the effects of external disturbances on vibrations. It is interesting to note that the imposition of external disturbance on the eccentricity-induced motion can improve the stability characteristics (Fig. 3h). Although instability is noticed near extremely low frequencies, i.e., $p=0$, external disturbances improve stability around the regions of $p=(K_I-1)/2$, 1, $(K_I+1)/2$, and K_I. This stabilization actually results from a favorable phase difference existing during the motion initiated by the vehicle pitching. Despite the important role played by the vehicle inertia ratio in determining critical damping, it exerts little influence on the width of the instabilizy zones.

Response of Flexural Vibration

Stable flexural response over 13 orbits for typical sets of parameters for gravity-stabilized satellites in circular and elliptic orbits are presented in Figs. 4 and 5, respectively, The values are nondimensionalized with respect to the initial modal displacement A_d and only impulsive disturbances have been considered in pitch plane with $d_2=0$. The plots are obtained using the analytical solutions derived earlier. The correctness of the solution is established by the numerical solution of the modal equation using the Runge-Kutta algorithm.

Figure 4a compares the low-frequency response to a small disturbance for three combinations of the inertia ratio and damping parameter. Reduction of the value of the inertia ratio by nearly 20 % alters the response only slightly. For the low-frequency response, a high value of damping is needed to damp the motion within a reasonable time. For high-frequency response (Fig. 4b), the damping is quite effective and the system is able to withstand much larger disturbances. Low-frequency behavior is, therefore, a source of greater concern.

Typical response in elliptic orbits are presented in Fig. 5. Since a higher degree of instability exists in eccentric orbits, vibration response is investigated only for small external disturbances of $c=0.15$. The gyroscopic nature of the libration introduced by the orbit ellipticity seems initially to distort the response. In the long run, the behavior is essentially similar to that predicted for the circular orbits. It may be noted that a large eccentricity (Fig. 5b) has appreciable influence on the response. It increases the magnitude and delays the occurrence of the maximum of modal amplitude. The effectiveness of damping is somewhat reduced in eccentric orbits. The modal response characteristics, such as the time period, magnification factor, logarithmic decrement of the flexural vibration, are not easily obtainable. For comparison, one case of high-frequency response (Fig. 5a) is included here. These results are also checked with the numerical solution of the modal equation.

Although only a few typical plots are included, they are sufficient to indicate the stable behavior of the system.

Conclusion

The present work is concerned with the analysis of in-plane flexural vibration of large, flexible, damped satellites undergoing pitching oscillations in circular and elliptic orbits. The analysis is directly applicable to beam- and plate-like and cable-connected space systems. The modal equations are found to be periodic for satellites moving in circular orbits and almost periodic for those moving in elliptic orbits. Successful application of perturbation techniques made it possible to obtain analytical expressions for stability boundaries and stable solutions of vibration. In the presence of structural damping, one finds that, although there are primarily two resonant frequencies in circular orbits, there are as many as six resonant frequencies in elliptic orbits. It is important to note that $p=1$ is a resonant frequency in the elliptic orbit and not in the circular orbit. An equally important conclusion is that the vehicle inertia ratio does not significantly influence the stability boundaries or the solution, but it is a crucial parameter in determining the critical damping.

Finally, a comment on multimode vibration is in order here. The actual deflection curve of the structure at any instant is due to the superposition of a finite number of modes taking part in the vibration. Therefore, the vibrational instability can be caused by the instability of any one of these modes, implying that in a given orbit any p should not lie in any of the unstable regions. It may be pointed out that, although the present solution for A_I has been shown to be exact solution of the approximate equation, one has to verify their applicability to multimode vibration based on an extensive numerical computation of the original set of hybrid differential equations obtained for specific structures.

The analysis presented here is applicable to a wide range of system parameters. The analytical expressions and the stability charts should prove valuable to designers of future large-scale spacecraft.

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